

The following resources related to this article are available online at www.sciencemag.org (this information is current as of November 16, 2009):

Updated information and services, including high-resolution figures, can be found in the online version of this article at:

<http://www.sciencemag.org/cgi/content/full/318/5855/1383b>

A list of selected additional articles on the Science Web sites **related to this article** can be found at:

<http://www.sciencemag.org/cgi/content/full/318/5855/1383b#related-content>

This article appears in the following **subject collections**:

Computers, Mathematics

http://www.sciencemag.org/cgi/collection/comp_math

Technical Comments

http://www.sciencemag.org/cgi/collection/tech_comment

Information about obtaining **reprints** of this article or about obtaining **permission to reproduce this article** in whole or in part can be found at:

<http://www.sciencemag.org/about/permissions.dtl>

Response to Comment on “Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture”

Peter J. Lu^{1*} and Paul J. Steinhardt²

Our study showed that both Gunbad-i Kabud and Darb-i Imam tessellations belong to a sequence of Islamic tilings that resolve into a common set of girih tiles, so local similarities are expected. However, historically accurate reconstructions show that Darb-i Imam is unique, the only known example that does not repeat periodically and that displays a self-similar transformation enabling its continuation ad infinitum to a perfect quasi-crystalline pattern.

For more than five centuries, medieval Islamic designers constructed a sequence of increasingly complex tessellations based on various close-packed arrangements of a common set of girih tiles, as we described in (1). Both the Gunbad-i Kabud (1197 C.E., Maragha, Iran) and the Darb-i Imam shrine (1453 C.E., Isfahan, Iran) belong to this sequence, so it is unsurprising that they share some local tile motifs, like the ones pointed out by Makovicky (2). However, although we agree that the Gunbad-i Kabud is of exceptional cultural value, it is nonetheless explicitly periodic and similar to a host of other periodic Islamic tilings discussed in (1). By contrast, the Darb-i Imam shrine stands out as the only example found thus far that is not part of a periodic pattern and that instead displays a self-similar subdivision into smaller tiles that can be continued ad infinitum to obtain an infinite, perfectly quasi-crystalline pattern.

Comparison of the two patterns is confounded by the fact that in (2) and in (3), Makovicky bases his analysis on drawn reconstructions that deviate from the actual historical record—a point Makovicky himself conceded in (3). These deviations blur the distinctions between the two patterns. To avoid this problem, we instituted

¹Department of Physics, Harvard University, Cambridge, MA 02138, USA. ²Department of Physics and Princeton Center for Theoretical Physics, Princeton University, Princeton, NJ 08544, USA.

*To whom correspondence should be addressed. E-mail: plu@fas.harvard.edu

the practice of overlaying our girih-tile reconstructions for both patterns over archival photos in our original paper [see figure 2, B and C, figure 3B, and figure S6 in (1)], to demonstrate their historical accuracy. Examination of these images reveals the differences between our and Makovicky’s interpretation.

First, the Gunbad-i Kabud decorative pattern is explicitly periodic: It seamlessly repeats every two panels, as shown in figure S6 in (1), and thus cannot be viewed as quasi-crystalline. By definition, a quasi-crystalline pattern cannot regularly repeat in any direction. In fact, the periodicity forces the regular repetition of a sequence of defects (which can be seen in the Gunbad-i Kabud pattern) relative to a perfect quasi-crystalline pattern, defects that cannot be removed by local tile rearrangements. By contrast, the far larger pattern on the Darb-i Imam shrine—the portion of the half arch in figure 3B in (1), which contains 213 complete girih tiles, versus 37 tiles in a single panel of the Gunbad-i Kabud shrine—does not repeat at all along the building.

Second, the Gunbad-i Kabud pattern does not display a self-similar subdivision into smaller replicas of the same tiles. There is a smaller underlying pattern, but it consists of totally different curved motifs, as shown in figure 2, C and D, in (1). By contrast, the self-similar subdivision is explicit at Darb-i Imam, as shown in figure 3, D and E, in (1). This difference is crucial: Repeated subdivision into smaller tiles is a standard mathematical technique for extending a finite quasi-

crystalline fragment such as the Darb-i Imam tiling (or a finite fragment of Penrose tiling) into an infinite quasi-crystalline pattern without periodic repetition.

Makovicky further claims that both the Gunbad-i Kabud and the Darb-i Imam tilings can be mapped precisely into Penrose’s tiling (3). The actual historic record does not support this claim either. It is simply not possible to map the periodic Gunbad-i Kabud pattern into the nonperiodic Penrose pattern. As for the Darb-i Imam shrine, we showed that the subdivision rule produces a quasi-crystalline tiling pattern that deviates systematically from Penrose’s (1). The Penrose tiling is just one of an infinite set of distinguishable, perfectly quasi-crystalline patterns with five-fold symmetry, and the Darb-i Imam pattern conforms to a different member of this same symmetry class.

In sum, although our study (1) acknowledged Makovicky’s contribution of bringing deserved attention to the Gunbad-i Kabud shrine and pointing out how fragments of it bear some relation to fragments of quasi-crystalline tilings, we cannot agree with his conclusions. The identification of tilings as Penrose or quasi-crystalline (or periodic, for that matter) must, by definition, be based on the symmetries and properties of the overall pattern, not just isolated fragments. To find fragments alone is not unique; we identified several examples of medieval Islamic tilings aside from Gunbad-i Kabud with equally impressive fragments that also belong to patterns that are periodic overall. Furthermore, if the goal is to understand the true historic development of mathematics and tessellations, it is critical to base the analysis on the unmodified historical records. In the present instance, the archival photos provide incontrovertible evidence that distinguishes the Gunbad-i Kabud and Darb-i Imam tiling patterns.

References

1. P. J. Lu, P. J. Steinhardt, *Science* **315**, 1106 (2007).
2. E. Makovicky, *Science* **318**, 1383 (2007); www.sciencemag.org/cgi/content/full/318/5855/1383a.
3. E. Makovicky, in *Fivefold Symmetry*, I. Hargittai, Ed. (World Scientific, Singapore, 1992), pp. 67–86.

23 July 2007; accepted 30 October 2007
10.1126/science.1146352